PROSPECTS OF THE NEW CENTURY

Dynamical stability of Hamiltonian systems

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Abstract Dynamical stability has become the center of study on Hamiltonian system. In this article we introduce the recent development in some areas closely related to this topic, such as the KAM theory, Mather theory, Arnold diffusion and non-singular collision of n-body problem.

Keywords: **dynamical stability, KAM theory, connecting orbits,**

Many mathematical models which describe problems in physics, astronomy and mechanics have the form of Hamiltonian equation,

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{q} = \frac{\partial H}{\partial p}, (p, q) \in R^{2n},$$

where $H = H(p, q, t)$ is called Hamiltonian, which, in general, is the energy of the system. A typical example of Hamiltonian system is the so called n-body problem in celestial mechanics, whose Hamiltonian has the form

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{1 \leqslant i \leqslant j \leqslant n} \frac{\mu}{|qi - q_j|}, \quad p_i = (p_i^1, p_i^2, p_i^3), q_i = (q_i^1, q_i^2, q_i^3).$$

Here $p_i = m_i \dot{x}_i$, $q_i = x_i$, and x_i, m_i are the position coordinate and the mass of the ith particle respectively.

The dynamical stability problem of Hamiltonian system has puzzled many mathematicians for a long time, and it has not been completely solved yet. A well-known example is the n-body problem. If r_{ij} is the distance between the ith and jth particles, people wish to know, along the trajectory in the phase space, whether or not the following quantity

$$\Delta = \max_{1 \leqslant i < j \leqslant n} \left\{ r_{ij}, \frac{1}{r_{ij}} \right\}$$

is bounded forever. Unfortunately, such a problem has remained unsolved. Even though there is no answer for this question, the general belief is that for $n \geqslant 3$, the set of orbits which escape to infinity or along which particles collide with each other is dense. We are not sure whether this set has positive Lebesgue measure.

Despite of many problems being unsolved, since the establishment of the basis for the study of modern dynamical systems by Poincare, the research on dynamical stability has made significant progress. And now researchers have acquired much deeper understanding of these questions than ever before.

1 KAM theory

In the early stage, people tried to solve a Hamiltonian system by finding its first integrals. For a Hamiltonian system with n degrees of freedom, if one can find n independent first integrals, which means that their Poinsson brackets are zero, then all solutions can be obtained by simple integration. Such a system is called integrable system. In this case a Hamiltonian system can be expressed by action-angle variables, that is, if $p \in R^n$ represents the action variable, $q \in T^n$ the angle variable, and the coordinate system of the phase space is $R^n \times T^n$, then the corresponding Hamiltonian depends only on action, i.e. $H = H(p)$. Therefore, the Hamiltonian equation has the following form:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = 0.$$

This gives $p = p_0$, $q = \omega t + q_0$ and $\omega = DH$, which indicate that the phase space is foliated by n-dimensional tori, each torus is invariant under the phase flow, therefore the phase flow is periodic or quasi-periodic. Obviously, the dynamical stability for integrable systems is very clear.

But integrable systems are rare, most of the systems in practice are not integrable, and there is a class of nonintegrable systems called near integrable. The coresponding Hamiltonian for this class could be expressed by $H = N(p) + P(p, q)$, where P is small. With his extraordinary perception, Kolmogorov realized that under small perturbation n-dimensional invariant tori cannot be destroyed completely. Actually, most of the invariant tori survive under small perturbation. Using a simple and classical idea of Newtonian iteration, he proposed a way of proving his claim. Later a complete proof was given independently by Arnold and Moser.

Their proof only requires that the frequency ω of quasi-periodic flow on torus satisfies the so-called Diophantus condition

$$|\langle \omega, k \rangle| = \left| \sum_{i=1}^{n} k_i \omega_i \right| \geqslant D\,|k|^{-\mu}, \qquad \forall\, k \in Z^n \setminus \{o\},$$

where D is a constant, and the system must satisfy Kolmogorov's nondegenerate condition, therefore small perturbation cannot destroy such a torus, but makes only small deformation on it.

Note that on R^n, for small enough constant D, the set of all $\omega \in R^n$ which satisfies corresponding Diophantus condition is a Cantor set, but with positive Lebesgue measure. As for $D \rightarrow 0$, the set has a full measure. Because of the Kolmogorov's nondegenerate condition, these n-dimensional invariant tori in phase space become a nowhere dense set, but with a positive Lebesgue measure.

KAM theory is of great importance. For any system of n degrees of freedom, it guarantees, in the sense of Lebesgure measure, the dynamical stability: if the perturbation is small enough, most of the motions of the given system are stable. With this conclusion we can explain that, at least in the sense of Lebesgue measure, the solar system is stable. Because the actions of the nine planets in the solar system with the Sun are much important than those with others, and the two-body problem is integrable, thus the solar system is a near integrable one.

For a system with 2 degrees of freedom, much more results could be derived by KAM theory. Since these systems do not depend on t, i.e. they are autonomous, the energy is preserved. The motions of the system in 4-dimensional phase space is restricted on a 3-dimensional energy surface. Since the invariant torus in this case is 2-dimensional, it divides the 3-dimensional surface into two disconnected parts, the motion starting from one part cannot reach the other and *vice versa*. Therefore the KAM theory completely solves the stability problem of this case. In celestial mechanics this explains why the motion of Trojan and Greek asteroids seems to be stable although they are almost at a critical position. In mathematics this theory guarantees Lyapunov stability of elliptic periodic points in the measure preserving twist maps.

After several years of efforts, the study on KAM theory itself has had a significant progress. The following are the most important archivements.

(i) Smoothness. The requirement on smoothness of the Hamiltonian H has been reduced. For a system of n degrees of freedom we only require continuous differentiability of order $2n+3$. For area preserving twist maps, the corresponding result is perfect now. It has been proven that for $H \in C^s(s > 3)$, the invariant circle is differentiable, but for $s < 3$ the invariant circle might be broken. This indicates that, for KAM theory, the requirement of smoothness is not the necessity of analysis, more or less it shows the difference in essence.

(ii) Kolmogorov's nondegeneracy. The condition of local diffeomorphism can be relaxed into the only requirement that the image of the frequency map should not be restricted into $(n-1)$-dimensional hyperplane. It should be mentioned that in n body problem the Kolmogorov's nondegenerate condition is not satisfied.

(iii) Lower-dimensional tori. The ideas of KAM theory could be used to find lower-dimensional invariant manifolds. Under the nondegenerate condition, if Diophanus and the first Melnokov condition are satisfied, the corresponding lower-dimensional invariant manifolds exist.

(iv) Co-dimension 1 invariant tori in volume preserving diffeomorphisms. In this situation the system has its dynamical stability.

(v) Lower-dimensional invariant manifolds in resonant parts. In resonant parts, n-dimensional tori disappear, but there exist lower-dimensional manifolds. Even though the measure of these manifolds might be zero, it has been predicted that these manifolds may control the complicated motion of the system. So far it is still not clear how many such lower-dimensional manifolds exist and what kind of the topological structure they have.

2 Mather theory

The disappearing of invariant torus clearly indicates the complicacy of dynamical behavior. In this respect, the two-dimensional measure preserving twist map, which corresponds to Hamiltonian systems of two degrees of freedom, are examples of relatively thorough studies. This map is generated by a function $h(x_1, x_2)$, $(x, y) \to (x_1, y_1)$ if

$$y = -\frac{\partial h}{\partial x}(x, x_1), \qquad y_1 = \frac{\partial h}{\partial x_1}(x, x_1).$$

Aubry and Mather independently studied the existence of quasi-periodic solutions. For any rotation number ω, Mather considered the following Lagrangian action:

$$F_\omega(\phi) = \int_0^1 h(\phi(t), \phi(t + \omega))\mathrm{d}t,$$

and proved that there exists a global minimal element ϕ satisfying the following properties:

(i) ϕ is increased monotonically, and $\phi(t+1) = \phi(t) + 1$.

(ii) $(\phi(t + \omega i), h_1(\phi(t + \omega i), \phi(t + \omega(i+1)))$ is an orbit of the map.

If there is no invariant circle, ϕ is discontinuous. In this case, if the rotation number is irrational, the graph of $(\phi(t + \omega i), h_1(\phi(t + \omega i), \phi(t + \omega(i+1)))$ is a Cantor set which can be embedded into a circle, the restriction of the system on this set is quasi-periodic motion with frequency ω.

By introducing Peierls barrier function

$$P_\omega(\xi) = \inf_{\substack{x \\ x_0 = \xi}} \sum h(x_i, x_{i+1}) - h(x_i^-, x_{i+1}^-), \qquad x_i^- = \phi(t + \omega i),$$

Mather proved that the necessary and sufficient condition for the existence of invariant circle is that the Peierls barrier function should be zero. If the function is not zero, there exist uncountable local, but not global, minimal elements for Lagrangian action. These local minimal elements correspond to Denjoy minimal sets. And these minimal sets cannot be embedded into a circle, but they can be embedded into a closed curve which goes around the cylinder more than once. The motion of the system on these Denjoy minimal sets is still quasi-periodic with rotation number ω.

Using this variational idea, Mather obtained the following extremely pretty result: inside the Birkhoff region of instability for any two global minimal sets $M_{\omega 1}$ and $M_{\omega 2}$ of Lagrangian action there is not only a connecting orbit, but also chaotic orbit in the following sense: for any countable minimal sets $(M_{\omega_1}, M_{\omega_2}, M_{\omega_3}, \cdots)$, there exists such an orbit which can be as close to these minimal sets as you want. The result of such kind can be generalized to any sequence of local minimal sets.

The efforts of looking for invariant sets as above can also be generalized to higher-dimensional space. Mather studied the higher-dimensional positive definite Lagrangian system, and proved that for any cohomoloy class ω there is an invariant measure M_ω such that its Lagrangian action is minimal. Furthermore it is a Lipschiz graph. Motion on this graph has frequency ω. So far the topological structure of this invariant measure has not been clear. And it has still not been known whether there is a connecting orbit between these invariant measures.

Based on the variational method used by Mather for two-dimensional twist map, Xia proposed an idea that could generalize Peierls barrier function into higher-dimensional space, which might be useful for connecting the orbits in higher dimensional cases.

3 Arnold diffusion

As mentioned above, there are fundamental differences between the systems with two degrees of freedom and those with higher degrees of freedom. For systems with two degrees of freedom, the two-dimensional invariant torus divides the three-dimensional energy surface into two disconnected parts, therefore these systems are dynamically stable in any sense. But when the degrees of freedom are more than two, the n-dimensional torus cannot divide $2n-1$ energy surface. In this case, the system may have an orbit which is far away from the initial position of action variable, or sometimes the orbits may spread all over the energy surface. Arnold constructed such an example. Consider a time-dependent Hamiltonian system with two degrees of freedom:

$$H_0 = \frac{1}{2}(p_1^2 + p_2^2), \qquad \varepsilon H_1 = \varepsilon(\cos q_1 - 1)(1 + \mu\beta), \qquad \beta = \sin q_2 + \cos t.$$

In general this system is considered as a system with two and a half degrees of freedom. He proved that the system has an orbit which connects two regions given by $p_2 < A$ and $p_2 > B$. Obviously this is a diffusive behaviour. This system has a special property that the other systems may not have: there exists a continuous family of hyperbolic two-dimensional tori, the corresponding action variable varies with order $O(1)$. By introducing two parameters β and ε, he made the stable and unstable manifolds of these tori intersect each other, so that it became a chain. The diffusive orbit was exactly along this chain. But, in general, a system may not have such a continuous family of hyperbolic lower-dimensional tori, and even so, the corresponding stable and unstable manifolds may not intersect, therefore it is hard to form a diffusive chain. Xia suggested that a possible diffusive path could be formed by bridging the stable and unsstable manifolds of Mather sets. To do this, we must have a clear understanding of the stable and unstable manifolds of higher-dimensional Mather sets. There is no doubt that this is an extremely challenging problem. So far, most studies have been on Arnold diffusion, but the systems in these studies are usually very special, so there are a lot of work to be done.

A closely related problem is the existence of lower-dimensional invariant manifolds in resonant area. According to Katok, the lower-dimensional manifolds are the remnants of broken higher-dimensional torus, which look like ghosts and affect the complicated motion of the system. We prove that, no matter what perturbation we choose, if it is small enough, there always exist some lower-dimensional tori in resonant part. These manifolds correspond not only to the minimal of Lagrangian action, but also to the minimax critical points of the action.

On 1998 International Conference of Mathematicians, Herman raised some unsolved problems on dynamical systems, in which the central question is about the dynamical stability: is there C^∞ Hamiltonian H in a C^2 neighborhood of $H_0 = \frac{1}{2} \| x \|$ such that the Hamiltonian flow on energy surface $H^{-1}(1)$ has a dense orbit?

4 Noncollison singularities

The study on n body problem has a long history. Since the Hamiltonian is inversely proportional to the distance between particles, when this distance is zero, the Hamiltonian reaches its singularity, which corresponds to the escaping of orbits in phase space. A century ago, Painleve raised the question of whether there exists noncollision singularity. He found that for three-body problem there is no noncollision singulairty, which indicates that such singularities may appear in the case of $n \geqslant 4$. After continual work of many mathematicians, this problem was finally solved by Xia. He constructed a five-body model. The first four particles can be seen as two binaries. The planes in which each binary is restricted are parallel to each other with certain distance, the segment connecting the two centers of masses is normal to the planes. The fifth particle moves back and forth along this segment. Under certain suitable conditions, the fifth particle accelerates constantly, and its velocity will approach infinity within finite time.

Besides its significance in dynamics itself, this result shows us a picture that the global singularity does appear in such a relatively simple system. General examples seem to give us an impression that singularity is always a local property.

References

1 Arnold, V. I., Instability of dynamical systems with several degrees of freedom. *Sov. Math. Dokl.*, 1964, 5: 581.
2 Aubry, S., Le Daeron, P. Y., The discrete Frenkel-Kontorova model and its extension, *Physica*, 1983, 8D: 381.
3 Berstien, D., Katok, A., Birkhoff periodic orbits for small perturbations of completely integrable Hamiltonians, *Invent. Math.*, 1987, 88: 224.
4 Bessi, U., An approach to Arnold diffusion through calculus of variations, *Nonlinear Ana.*, 1996, 26: 1115.
5 Bourgain, J., Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Shrödinger equations, *Annals of Mathematics*, 1998, 145.
6 Cheng, C. Q., Sun, Y. S., Existence of invariant tori in three dimensional measure preserving mappings, *Celest. Mech.* 1990, 45: 275.
7 Cheng, C. Q., Sun, Y. S., Existence of KAM tori in degenerate Hamiltonian systems, *J. Diff. Eqns.*, 1994, 114: 288.
8 Cheng, C. Q., Birkhoff-Kolmogorov-Arnold Moser tori in convex Hamiltonian systems, *Commun. Math. Phys.*, 1996, 177: 592.
9 Cheng, C. Q., Lower dimensional invariant tori in the regions of instability for nearly integrable Hamiltonian systems, *Commun. Math. Phys.*, 1999, 203: 385.
10 Cheng, C. Q., Cheng, W., Connecting orbits among Denjo sets in twist area-preserving diffeomorphisms, 1999, 155: 311.
11 Eliasson, L. H., Perturbations of stable invariant tori for Hamiltonian systems, *Ann. Sc. Norm. Pisa*, 1998, 15: 115.
12 Herman, M., Some open problems in dynamical systems, in *Proceedings of ICM*'98, 1998, 2: 797.
13 Herman, M., Sur les courbes invariantes pae les diffeomorphisms de l'anneau *Asterisque*, 1983, 103, 1986, 144.
14 Katznelson, Y., Ornstein D., A new method for twist theorems, *J. Ana. Math.*, 1993, 60: 157.
15 Mather, J., Existence of quasi-periodic orbits for twist homeomorphisms of the annulus, *Topology*, 1982, 21: 457.

16 Mather, J., More Denjoy minimal sets for area preserving diffeomorphisms, *Comment. Math. Helv.*, 1985, 60: 508.

17 Mather, J., Action minimizing invariant measures for positive definite Lagrangian systems, *Math. Z.*, 1991, 207: 169.

18 Mather, J., Variational construction of connecting orbits for twist diffeomorphisms, *J. AMS*, 1991, 4: 207.

19 Moser, J., Dynamical systems-past and present, in *Proceedings* of *ICM*'98, 1998, 1: 381.

20 Russmann, H., Invariant tori in the perturbation theory of weakly non-degenerate integrable Hamiltonian systems, Preprint of University of Mainz (1998). to appear *Ann. Sci. Ecole Norm.* Sup.

21 Salamon, D., Zehnder, E., KAM theory in configuration space, *Comment. Math. Helv.*, 1989, 64: 39.

22 Xia Z. Arnold diffusion: a variational construction, in *Proceedings* of *ICM*'98, 1998, 2: 867.

23 Xia, Z. The existence of noncollision singularities in Newtonian systems, *Annals Math.* 1992, 135: 411.

24 Xu, J., You, J., Qui, Q., Invariant tori for nearly integrable Hamiltonian systems with degeneracy, *Math Z.* 1997, 226: 375.

25 Yoccoz, J., Travaux de Herman sur les tores invariants, *Asterisque*, 1992, 206: 311.

26 Yoccoz, J., Recent developments in dynamics, in *Proceedings* of *ICM*'94, 1995, 246—265.